Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(00)00076-9
J. Appl. Maths Mechs, Vol. 64, No. 4, pp. 509–519, 2000

0021-8928/00/$—see front matter

ANALYTICAL SYNTHESIS OF TIME-OPTIMAL CONTROL IN A THIRD-ORDER SYSTEM†

L. D. AKULENKO and G. V. KOSTIN

Moscow

(Received 29 October 1999)

A time-optimal control that steers the phase point for a third-order linear system to the origin is constructed in an explicit analytical form. It is assumed that the characteristic exponents are zero, and the constraints on the control function are non-symmetric. The system simulates the dynamics of a point mass driven by a force whose rate of change can be regulated. An optimal control is constructed both in the feedback and open-loop forms. In the latter case, the optimal control is a function of time. Relations are derived for the switching curve and surface and for the time intervals of the motion; optimal phase trajectories are constructed; the feedback control portrait is investigated. The influence of a parameter characterizing the degree of asymmetry of the constraints is studied. "Near-optimal" control modes, which are much simpler to implement, are constructed. © 2000 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

We consider a time-optimal control problem [1] for a third-order controlled system.

$$\begin{aligned}&\dddot{x}=u;\quad x(0)=x^0,\quad \dot{x}(0)=\dot{x}^0,\quad \ddot{x}(0)=\ddot{x}^0\\&x(t_f)=\dot{x}(t_f)=\ddot{x}(t_f)=0\\&t_f\to\min_u,\quad u^-\leqslant u\leqslant u^+,\quad u^-<0,\quad u^+>0\end{aligned}\tag{1.1}$$

It is required to find an optimal control u in the form of a programme $u^*=u_p(t,x^0,\dot{x}^0,\ddot{x}^0)$ and a synthesis $u^*=u_s(x,\dot{x},\ddot{x})$, as well as the optimal response time of the motion $t_f^*=T(x^0,\dot{x}^0,\ddot{x}^0)$, the switching times of the control, which is of the bang-bang type, the Bellman function $T(x,\dot{x},\ddot{x})$ of problem (1.1) and optimal trajectories $x=x^*(t,x^0,\dot{x}^0,\ddot{x}^0)$, $\dot{x}=\dot{x}^*(t,x^0,\dot{x}^0,\ddot{x}^0)$, $\ddot{x}=\ddot{x}^*(t,x^0,\dot{x}^0,\ddot{x}^0)$; in other words, the problem is to construct an optimal feedback control portrait [1]. No complete solution of this problem is known in scientific literature. It is of certain methodological and applied interest.

We note that at an early stage, when the mathematical technique of the maximum principle was first established, schemes for constructing the switching surface, based on Fel'dbaum's theorem, were developed ([2, 3], etc.), and equations to determine the time intervals and simplified relations were presented [3]. Problem (1.1) has been investigated [4] for the case of symmetrical constraints $(-u^-=u^+)$ when no terminal condition is imposed on the quantity $\ddot{x}(t_f)$. Krasovskii's methods of the moment problem [5] have been used to analyse the case of an equation of arbitrary order $x^{(n)}=u$, $|u|\leqslant 1$, when $n=1,2,3,\ldots$, with conditions of type (1.1) $x(t_f)=\dot{x}(t_f)=\ldots x^{(n-1)}(t_f)=0$. General relations have been derived for the minimum time and for the singular sets (switching curves, surfaces and hypersurfaces of the control), in particular, for $n=3$, but the feedback control portrait has not been investigated.

A solution of the control problem (1.1) exists for arbitrary values of $x^0,\dot{x}^0,\ddot{x}^0$ (see Section 4). A time-optimal control satisfies necessary and sufficient conditions in the form of the maximum principle [1]. For greater convenience in applications, we will represent the third-order equation (1.1) in the form of the system.

$$\begin{aligned}&\dot{w}=a+u,\quad \dot{v}=w,\quad \dot{x}=v;\quad w(0)=w^0,\quad v(0)=v^0,\quad x(0)=x^0\\&w(t_f)=v(t_f)=x(t_f)=0;\quad t_f\to\min_u,\quad |u|\leqslant 1\\&a=(u^-+u^+)/(u^+-u^-),\quad |a|<1\ \ (2\dddot{x}/(u^+-u^-)\to\dddot{x})\end{aligned}\tag{1.2}$$

†*Prikl. Mat. Mekh.* Vol. 64, No. 4, pp. 532–544, 2000.

Proceeding in the standard way, we introduce variables r, q, p adjoint to w, υ, x and define the form of the optimal control on the basis of the maximum condition for the Hamiltonian H

$$\begin{aligned}
&u^* = \operatorname{sign} r, \quad r = r^0 - q^0 t + p^0 t^2/2 \\
&t \geq 0, \quad r^0, q^0, p^0 = \text{const} \\
&H^* = H\big|_{u=u^*} = |r| + ra + qw + pv = \text{const} \geq 0 \\
&\dot{p} = 0, \quad \dot{q} = -p, \quad r = -q
\end{aligned} \tag{1.3}$$

It follows from (1.3) that the optimal control u^* is of the bang-bang type and may have two, one or no switching points (Fel'dbaum's theorem [1]), depending on the values of w^0, υ^0, x^0. The unknown constants r^0, q^0, p^0 and the optimal time t_f are determined by three boundary conditions and one normalization condition as a function of the initial (measured) data (w^0, υ^0, x^0). This approach, however, is extremely cumbersome and difficult to implement.

By analogy with the two-dimensional feedback control portrait for the equation $\ddot{x} = a + u$ [1], we will construct sets in the phase space (w, υ, x) corresponding to the number of switching points. According to the maximum principle, this space is divided by the switching surface into two parts. On the surface there is a switching curve with the property that if the phase point moves along that curve, it will reach the origin without switchings. We will find analytical expressions describing the different parts of the motion and the sets just described. Introduction of a Cartesian system of coordinates WVX yields a readily visualized geometrical representation of the constructions (see Fig. 1). We will refer to the WV plane as horizontal.

At the first stage $0 \leq t \leq \vartheta$ the point $Q = (w, \upsilon, x)$ moves from an arbitrary initial state $Q^0 = (w^0, \upsilon^0, x^0)$ (1.2), with the corresponding value of $u = \pm 1$, until it reaches the surface P, on which the control is switched. Thus we have the following expressions when $t = \vartheta$

$$\begin{aligned}
&w^0 + \chi_1 = w_1, \quad v^0 + w^0\vartheta + \chi_2 = v_1 \\
&x^0 + v^0\vartheta + w^0\vartheta^2/2 + \chi_3 = x_1, \quad \chi_k = (a+u)\vartheta^k/k!, \quad k = 1,2,3 \\
&\vartheta \geq 0, \quad Q^0 = (w^0, v^0, x^0) \in R, \quad Q_1 = (w_1, v_1, x_1) \in P
\end{aligned} \tag{1.4}$$

After the control changes sign, $u \to -u$, the phase point moves along the surface P from state $Q_1 = (w_1, \upsilon_1, x_1)$ until it intersects the switching curve L, which divides the surface P into two parts. At the second step, $\vartheta \leq t \leq \vartheta + \tau$, the phase portrait is two-dimensional (similar to that of the classical case [1]); finally, we obtain expressions of the same type as (1.4) when $t = \vartheta + \tau$

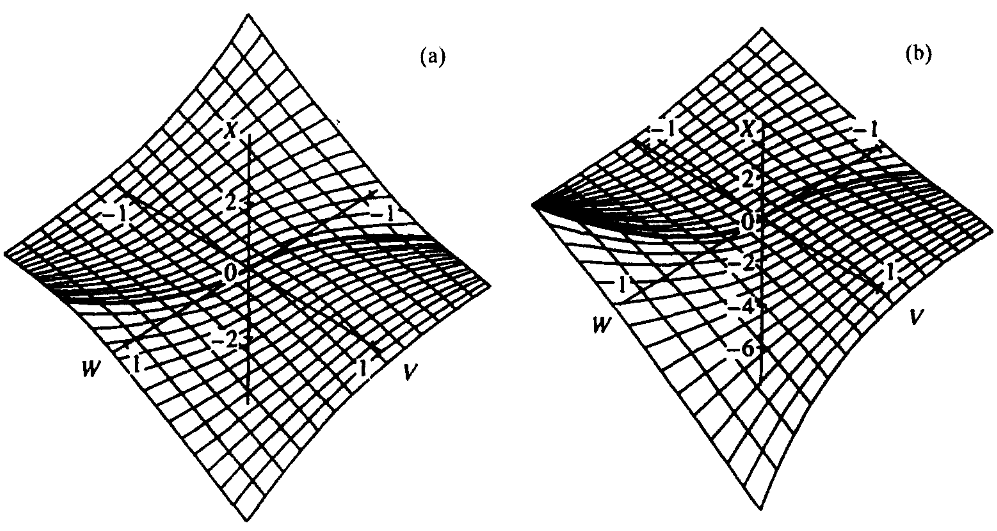

Fig. 1.

$$w_1 + \chi_1 = w_2, \quad v_1 + w_1\tau + \chi_2 = v_2$$
$$x_1 + v_1\tau + w_1\tau^2/2 + \chi_3 = x_2 \tag{1.5}$$
$$\chi_k = (a-u)\tau^k, \quad k = 1, 2, 3; \quad \tau \geqslant 0, \quad Q_2 = (w_2, v_2, x_2) \in L$$

The control changes sign again on the switching curve L, and the phase point Q moves along the curve from state Q_2 to the origin $Q_f = (0, 0, 0)$ in the time $\theta \geqslant 0$; when $t = t_f = \vartheta + \tau + \theta$ we have the following relations

$$w_2 + \chi_1 = 0, \quad v_2 + w_2\theta + \chi_2 = 0 \tag{1.6}$$
$$x_2 + v_2\theta + w_2\theta^2/2 + \chi_3 = 0,$$
$$\chi_k = (a+u)\theta^k/k!, \quad k = 1,2,3; \quad \theta \geqslant 0$$

Relations (1.4)–(1.6) are parametric equations of the switching surface P and switching curve L of the bang-bang control; they also define the intervals of the motion ϑ, τ, θ and the optimal time $t_f = \vartheta + \tau + \theta$ as functions of an arbitrary initial point Q^0. In addition, the conditions $\vartheta, \tau, \theta \geqslant 0$ uniquely define the optimal control both at the beginning and throughout the entire process of the controlled motion.

Thus, it is required to solve the equations for the unknown characteristics, to investigate their properties and to construct a portrait of the time-optimal motion of the phase point Q from the position Q^0 to the terminal state Q_f. If the point Q^0 is fixed, relations (1.4)–(1.6) define an optimal open-loop control: the switching times and the initial value of $u = \pm 1$.

2. CONSTRUCTION OF THE OPTIMAL FEEDBACK CONTROL

We will first use relations (1.4)–(1.6) to determine the portrait of the time-optimal motion in the entire phase space: $Q^0 \to Q \in R$. To that end, we construct the switching curve L and the switching surface P of the bang-bang control.

2.1. The switching curve. The switching curve is parametrically defined ($\theta \geqslant 0$ is the parameter) by Eqs (1.6); it consists of two optimal trajectories reaching the origin. We solve Eqs (1.6) for w_2, v_2, x_2; we obtain the desired expressions

$$W = w_2 = -\chi_1, \quad V = v_2 = \chi_2, \quad X = x_2 = -\chi_3 \tag{2.1}$$
$$\chi_k = (a+u)\theta^k/k!, \quad k = 1,2,3$$
$$u = u_2 = \pm 1, \quad \theta \geqslant 0, \quad L = L_+ \cup L_-$$

It follows from (2.1) that if $Q \in L$, that is, some $\theta > 0$ exists for which the vector equation $Q = Q_2(\theta)$ is satisfied, the optimal control is $u_2 = -\operatorname{sign} w_2$ and motion occurs along the corresponding branch $L_\pm$ of the curve L. The projections of the switching curve onto the WV and WX planes have the following form (see Figs 2 and 3, the dashed curves 1, 2 are for $a = 0$ and $a = 1/2$, respectively)

$$V = \xi_2 W^2 = -(1 - a\operatorname{sign} W)^{-1}|W|W/2$$
$$X = \xi_3 W^3, \quad \xi_k = (a - \operatorname{sign} W)^{1-k}/k!, \quad k = 2,3 \tag{2.2}$$
$$(V = (9/2)^{1/3} u_2 (1 + au_2)^{1/3} X^{2/3}, \quad u_2 = -\operatorname{sign} X$$
$$X = -(2/9)^{1/2} u_2 (1 + au_2)^{-1/2} |V|^{3/2}, \quad u_2 = \operatorname{sign} V$$

The curves $V(W)$ and $X(W)$ of (2.2) have horizontal tangents (the W axis is horizontal).

It follows from (2.1) and (2.2), in particular, that $u = u_2 = \pm 1$ for $X, X, W \lessgtr 0$, that is, the control u is completely defined on the curve L. Note that if $a = 0$, the curves (2.2) are invariant under the substitution $u \to -u$, $Q \to -Q$. The curve L divides the surface P into two parts $P_\pm$ corresponding to $u_2 = \pm 1 (L \notin P_\pm, P = P_+ \cup P_- \cup L)$; see below.

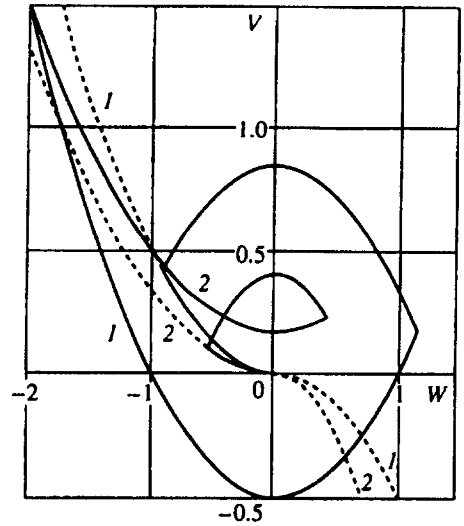

Fig. 2.

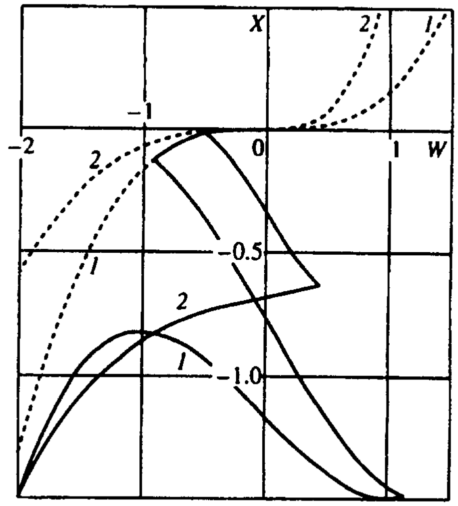

Fig. 3.

2.2. *The switching surface*. The construction of the switching surface P is based on relationships (1.5) and (1.6), by analogy with (2.1) and (2.2). The parametric representation is

$$\begin{gathered} W = w_1 = -\chi_1, \quad V = v_1 = \chi_2, \quad X = x_1 = -\chi_3 \\ \chi_k = a(\theta+\tau)^k - u_1[(\theta+\tau)^k - 2\tau^k]/k!, \quad k = 1,2,3 \\ \theta, \tau \geqslant 0, \quad u_1 = \pm 1 \end{gathered} \tag{2.3}$$

Taking into consideration that by (2.3) θ and τ are non-negative, we obtain the desired representations and the condition to be imposed on W and V

$$\begin{gathered} \theta = (1+au_1)^{-1}[(1+au_1)W^2/2 + u_1(1-a^2)V]^{1/2} \\ \tau = u_1(1+au_1)^{-1}W + (1+au_1)(1-au_1)^{-1}\theta, \quad u_1 = \pm 1 \\ u_1(1-a^2)V + (1+au_1)W^2/2 \geqslant 0 \end{gathered} \tag{2.4}$$

Substitution of (2.4) into the third relation in (2.3), for X, yields an expression $X(W, V, u_1)$ in which the function $u_1(Q_1)$ is as yet undetermined. It follows from the necessary and sufficient optimality conditions of the maximum principle [1] that the inequality in (2.4) defines the domains of the values of W, and V for which $u_1 = \pm 1$, respectively

$$u_1 = u_1^* = \operatorname{sign} W \tag{2.5}$$

$$(1-a^2)V \operatorname{sign} W \geqslant -(1 + a \operatorname{sign} W)W^2/2$$

Thus, the explicit expression for the switching surface P becomes

$$\begin{gathered} X = u_1\tau^3/3 + (1+au_1)^{-2}(W - 2u_1\tau)^3/6 \equiv X^*(W, V, u_1) \\ \tau = (1-au_1)^{-1}(u_1 W + [(1+au_1)W^2/2 + u_1(1-a^2)V]^{1/2}) \equiv \tau^*(W, V, u_1) \\ u_1 = \pm 1, \quad V \gtrless -(1 + a \operatorname{sign} W)(1-a^2)^{-1}\,|W|\,W/2 \equiv V^*(W) \end{gathered} \tag{2.6}$$

According to (2.5) and (2.6), the function $X^*(W, V, u_1)$ is smooth everywhere except for the set of the values of W and V corresponding to the change of sign of u_1

$$V = V^*(W) = -(1-a^2)^{-1}(1 + a \operatorname{sign} W)\,|W|\,W/2 \tag{2.7}$$

A direct check will show, after (2.7) is substituted into (2.6), that the curve along which the two smooth parts of the surface P join together is the switching curve L of (2.2). Indeed, we have

$$(1 \mp 1)W/2 + (1 \mp a)\tau^*(W, V^*(W), \pm 1) = \begin{cases} W, & W \geqslant 0 \\ 0, & W \leqslant 0 \end{cases} \tag{2.8}$$
$$X(W, V^*(W), \pm 1) = (1 \mp a)^{-2} W^3/6$$

The system of equations (2.7) and (2.8) defines the switching curve L of (2.2).

We have thus established that the switching surface is $P = P_+ \cup L \cup P_-$, where the parts $P_\pm$ of the surface correspond to the values $u_1 = \pm 1$, respectively. The analytical expressions for $P_\pm$ have the form

$$P_\pm = \{W, V, X: \quad X = X^*(W, V, u_1), \quad u_1 = \pm 1, \quad V \gtrless V^*(W)\} \tag{2.9}$$

As a result, we have constructed singular sets: the switching surface P and switching curve $L \subset P$ of the bang-bang control $u(Q)$. These sets also depend on the parameter a, $|a| < 1$. They are shown graphically in Figs 1–3; for clarity, we show the situation for $a = 0$ and $a = 1/2$, respectively. Figure 1 is an isometric projection of the switching surface P and the switching curve L of the control. The orthogonal projections of the switching curves onto the WV and WX planes are shown in Figs 2 and 3 (dashed curves 1 and 2) for $a = 0$ and $a = 1/2$, respectively.

2.3. Synthesis of the time-optimal feedback control. The bang-bang control is constructed using expressions (2.1)–(2.6) and (2.9). In the regular case, where $Q \notin P$, the control u^* is defined as follows:

$$u^* = u_0^*(Q) = \pm 1, \quad x \lessgtr X(w, v) \quad (Q \in R_\pm)$$
$$R_\pm = \{w, v, x: x \lessgtr X^*(w, v, u_1)\} \tag{2.10}$$

The control $u_0(Q)$ (2.10) steers the phase point Q onto the switching surface P or, more precisely, onto one of its parts $P_\pm$, on which the optimal control changes in sign (when there are no perturbations). The phase point $Q = Q_1 \notin P_\pm$ then moves along the surface P and reaches the curve L: $Q_1 = Q_2 \notin L_\pm$. After the control has changed sign, the point Q_2 moves along the curve $L_\pm$ to the terminal point Q_f.

If the system is subject to uncontrolled perturbations, the synthesis of a time-optimal feedback control involves verification that the phase point belongs to the sets $R_\pm$, $P_\pm$, $L_\pm$, that is

$$u^* = u_s(Q) = \pm 1, \quad Q \in R_\pm \cup P_\pm \cup L_\pm \tag{2.11}$$

at each instant of time.

2.4. Construction of the optimal trajectory. Simulation of the motion of the unperturbed system (1.2), which is optimally controlled according to the feedback law (2.11), reduces to elementary integration of the equations for the piecewise-constant values of the function $u = u_s(Q)$. At the first step we have

$$w(t) = w^0 + \chi_1, \quad v(t) = v^0 + w^0 t + \chi_2, \quad x(t) = x^0 + v^0 t + w^0 t^2/2 + \chi_3$$
$$\chi_k = (a + u_0^*(Q^0))t^k/k!, \quad k = 1, 2, 3, \quad 0 \geqslant t \geqslant \vartheta \tag{2.12}$$

where $Q^0 \notin P$ (the regular case) and the control $u_0^*(Q^0)$ is chosen on the basis of (2.10)

$$u_0^* = -\operatorname{sign}(x^0 - X(w^0, v^0, u_1))$$

After the surface P has been reached at a certain time $t_1 = \vartheta$ for which

$$x(\vartheta) = X(w(\vartheta), v(\vartheta), u_1)$$

the control changes sign; note that when that happens $Q(\vartheta) \notin L$. The phase point moves on the part of the surface P_+ or P_- in accordance with the formulae

$$\begin{aligned}
&w(t)=w_1+\chi_1,\quad v(t)=v_1+w_1(t-\vartheta)+\chi_2\\
&x(t)=x_1+v_1(t-\vartheta)+w_1(t-\vartheta)^2/2+\chi_3\\
&\chi_k=(a-u_0^*(Q^0))(t-\vartheta)^k/k!,\quad k=1,2,3
\end{aligned}\tag{2.13}$$

By (2.13), at some time $t = t_2 = \vartheta + \tau$ the phase point intersects the curve L, that is

$$\begin{aligned}
&v(t_2)=-\chi_2\,|w(t_2)|\,w(t_2),\quad w(t_2)=w_2,\quad v(t_2)=v_2\\
&x(t_2)=\chi_3 w^3(t_2)=x_2\\
&\chi_k=(1-a\operatorname{sign}w(t_2))^{1-k}/k!,\quad k=2,3
\end{aligned}\tag{2.14}$$

that is, $Q(t_2) = Q_2 \in L$. Further motion of the point $Q(t)$ for $t_2 < t \leqslant t_f = t_2 + \theta$ occurs, by analogy with (2.12) and (2.13), from the point Q_2 (2.14) to the origin Q_f under the control $u = u_0^*(Q^0)$. By (1.6), the terminal point Q_f is reached after a time interval $\theta = (1 - a\ \mathrm{sign} w_2)^{-1}|w_2|$. Note that if $Q = Q_1 \in P_\pm$, the time intervals τ, θ are determined similarly, on the basis of the values w_1, v_1, using formulae (2.3)–(2.7); see below.

Under real conditions, the system may experience perturbations, either constantly acting or impulsive with respect to w. This leads to singular control modes sliding along the surface P and curve L. The control $u_s^*(Q)$ is chosen theoretically by (2.11) at each instant of time (in practice – fairly frequently, for example, at times separated by the basic time step of the integration or measurement process of the phase vector (Q). An example – computation of the optimal phase trajectory for $a = 0$ and $a = 1/2$ (solid curves 1 and 2) – projected onto the WV and WX planes, is shown in Figs 2 and 3 for initial values of the phase variables $w^0 = -2$, $v^0 = 1.5$, $x^0 = -1.5$; the optimum response times are $t_f = 6.15$ and $t_f = 3.92$, respectively, see below.

3. DETERMINATION OF THE TIME CHARACTERISTICS OF THE MOTION IN THE TIME-OPTIMAL PROBLEM

A synthesis of the feedback control was constructed in Section 2; it is based on measuring the phase vector and may be implemented without computing the relevant time intervals ϑ, τ θ and the minimum value of $t_f = \vartheta + \tau + \theta$. However, effective determination of these quantities is extremely important from both the theoretical and applied viewpoints, e.g. in order to construct an open-loop control and optimal trajectories according to Section 2.4. This may be done using formulae (1.4)–(1.6).

3.1. Determination of the time intervals. Let us eliminate the unknowns Q_1 and Q_2 from (1.4)–(1.6). We obtain an algebraic system for the required ϑ, τ and θ

$$\begin{aligned}
&\Delta\zeta=-\chi_1,\quad \Delta\eta=\chi_2,\quad \Delta\xi=-\chi_3;\quad t_f=\vartheta+\tau+\theta\\
&\Delta\zeta=(w^0+\lambda_1)/u,\quad \Delta\eta=(v^0-\lambda_2)/u,\quad \Delta\xi=(x^0+\lambda_3)/u\\
&u=u_0^*\ (Q^0)=\pm 1\\
&\chi_k=[t_f^3-2(\tau+\vartheta)^k+2\vartheta^k]/k!,\quad \lambda_k=at_f^k/k!,\quad k=1,2,3
\end{aligned}\tag{3.1}$$

The value of u in (3.1) is chosen in accordance with (2.10) or (2.11). Solving the first two equations for τ and ϑ in terms of $\Delta\zeta$, $\Delta\eta$ and the unknown t_f, we obtain

$$\begin{aligned}
&\tau_*=(t_f+\Delta\zeta)/2,\quad \vartheta_*=(t_f-\Delta\zeta)/2-\theta_*=\Phi_\vartheta(t_f)/\tau_*\\
&\theta_*=\Phi_\theta(t_*)/\tau_*\\
&\Phi_{\theta,\vartheta}=\pm(4\Delta\eta\mp\Delta\zeta^2+2\Delta\zeta t_f\pm t_f^2)/8,\quad \vartheta_*+\tau_*+\theta_*\equiv t_f
\end{aligned}\tag{3.2}$$

Note that, by (3.1), the values of $\Delta\zeta$ and $\Delta\eta$ in (3.2) depend on the unknown t_f. Substituting the algebraic functions τ_* and ϑ_* (or τ_* and θ_*, $\theta_* = t_f - \tau_* - \vartheta_*$) (3.2) into the relation for $\Delta\zeta$ (3.1) and multiplying by $\tau_* > 0$ (in the regular case, where $Q^0 \notin L$), we obtain a fourth-order equation for the unknown t_f, for example, in the form

$$F(t_f,\Delta\zeta,\Delta\eta,\Delta\xi)=\Delta\xi\tau_*+t_f^3\tau_*/6-t_f^2\tau_*^2+2t_f\Phi_\theta\tau_*+t_f\tau_*^3-\Phi_\theta\tau_*^2-\tau_*^4/3-\Phi_\theta^2=0 \tag{3.3}$$

The functions τ_*, Φ_θ, $\Delta\xi$ (and Φ_θ, $\Delta\zeta$, $\Delta\eta$) are determined by (3.1) and (3.2) as functions of the unknown t_f, the given quantities w^0, υ^0, x^0, and the parameter a. We have to determine the minimum root t_f^* of Eq. (3.3) which satisfies the conditions

$$\begin{gathered} t_f^*=t_f(Q^0,a)>0,\ \ \tau^*=\tau(Q^0,a)>0 \\ \vartheta^*=\vartheta(Q^0,a)>0,\ \ \theta^*=\theta(Q^0,a)>0 \end{gathered} \tag{3.4}$$

As in the case of symmetrical constraints [3], it turns out that these conditions hold for the maximum root of Eq. (3.3).

The desired root t_f^* of (3.4) may be determined analytically using the Cardano formula or numerically for fixed Q^0 and a. It is fairly simple to estimate the minimum value t_f^- and maximum value t_f^+ ($t_f^+\geqslant t_f^->0$) of t_f, in particular, using the "near-optimal" approach to the solution of the problem (see Section 4).

Note that if $Q^0 \notin P$, all the above time intervals are strictly positive. If $Q^0\in P$, but $Q^0\notin L$, then $\vartheta_*=0$, τ_*, $\theta_*>0$. Finally, if $Q^0\in L$, but $Q_0\neq Q_f$, we have $\tau_*=\vartheta_*=0$, $\theta_*>0$.

For the figures specified above, the time intervals for $a=0$ are as follows: $t_f^*=6.15$, $\vartheta_*=3.16$, $\tau^*=2.08$, $\theta^*=0.92$; similarly, if $a=1/2$, the desired quantities are $t_f^*=3.92$, $\vartheta_*=1.61$, $\tau^*=1.94$, $\theta^*=0.3$.

3.2. Construction of the Bellman function. The optimal response time t_f^* for an arbitrary point $Q\in R$ – the Bellman function $T(Q)$ of problem (1.2) – is determined as a positive piecewise-smooth solution of the Cauchy problem for the Hamilton–Jacobi–Bellman equation

$$\begin{gathered} -|\partial T/\partial w|+a\partial T/\partial w+w\partial T/\partial v+v\,\partial T/\partial x=-1 \\ u^*=-\partial T/\partial w\,|\partial T/\partial w|^{-1},\ \ T(Q)>0,\ \ T(Q_f)=0 \end{gathered} \tag{3.5}$$

The relation between dynamic programming and the maximum principle was discussed in [1]. To solve the Cauchy problem (3.5), one has to construct the equations of characteristics and solve the two-point boundary-value problem of the maximum principle, in the form (1.2), (1.3). The corresponding expressions (3.2)–(3.4) for an arbitrary point Q^0 determine the desired Bellman function $T(Q)=t_f^*(Q)$, which depends on the three variables w, υ, x and on the parameter a. It may be defined as a computational procedure or represented by sections $t_f^*=$ const, that is, two-dimensional level surfaces in the three-dimensional space of the variables w, υ and x for fixed a.

4. "NEAR-OPTIMAL" CONTROL MODES

Besides the optimal control modes just constructed, comparatively simple methods may be proposed that do not require very much more time. They are analogous to "coordinatewise descent," and their implementation does not require large computational resources.

4.1. The simplest method of control. This mode consists of three stages: (1) time-optimal steering to zero acceleration w; (2) time-optimal steering to zero velocity υ and to acceleration w from the state $w=0$; (3) time-optimal steering to zero value of the coordinate x, to velocity υ and to acceleration w from the state $w=0$, $\upsilon=0$. Thus, the phase point first moves in the VX plane, then along the X axis, finally reaching the terminal point Q_f. The corresponding controls $u(t)$ are constructed using piecewise-constant (bang-bang) functions, which become Walsh functions if $a=0$ [7]. Figure 4 is a schematic representation of the control process.

Thus, at the first step we have the following expressions (in the regular case position, $Q^0\notin L$)

$$\begin{aligned} &0\leqslant t\leqslant t_1=\tau_w\equiv|w^0|/d^-(w)^0,\ \ d^\pm(w)\equiv 1\pm a\operatorname{sign}w \\ &u=u_w(t)\equiv-\operatorname{sign}w^0(h(t)-h(t-\tau_w)) \\ &Q(t_1)=Q_1=(0,v_1,x_1),\ \ v_1=v^0+(1-d^-(w^0)/2)|w^0|w^0 \\ &x_1=x^0+v^0|w^0|+(3-d^-(w^0))w^{03}/6 \end{aligned} \tag{4.1}$$

where h is the Heaviside step function.

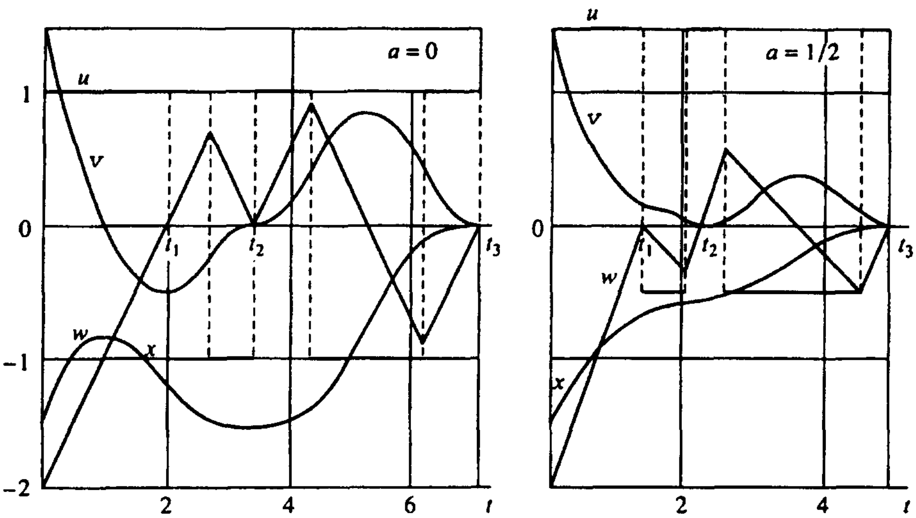

Fig. 4.

It follows from (4.1) that $\tau_w = |w^0|$ for $a = 0$ (symmetrical constraints, $u^- = u^+$), since $d^\pm = 1$. In addition, the ratio $k_w = \tau_w/|w^0|$ is a monotone decreasing or increasing function of a as $|a| \to 1$, and, moreover, $k_w \to 1/2$ or $k_w \to \infty$, since $d^\pm \to 2$ or $d^\pm \to 0$. The expressions for υ_1 and x_1 in the special case of symmetrical constraints are obtained from (4.1) by setting $a = 0$.

By (4.1), the acceleration w is a linear function of t; the velocity υ is described by a segment of a parabola and the coordinate x by a segment of a cubic parabola. The quantities υ_1 and x_1 may be arbitrary; compared with $|\upsilon^0|$ and $|x^0|$, their absolute values at the end of the control process may be increased or decreased (see Fig. 4). It is obvious that the value of τ_w (4.1) is a lower bound t_f^- for the minimum time t_f (see Section 3.1).

If $\upsilon_1 \neq 0$ (the second stage), the control and the other characteristics of the process are defined by the relations.

$$
\begin{aligned}
& t_1 < t \leqslant t_2 = \tau_w + \tau_v, \quad \tau_v = |v_1|^{1/2}(\nu + \nu^{-1}) \equiv \delta + \sigma \\
& \nu = (d^+(v_1)/d^-(v_1))^{1/2}, \quad Q(t_2) = Q_2 = (0, 0, x_2) \\
& u_v \equiv -\operatorname{sign} v_1[h(t - t_1) - 2h(t - t_1 - \delta) + h(t - t_1 - \tau_v)] \\
& x_2 = x_1 + v_1\tau_v - \operatorname{sign} v_1[d^-(v_1)(\delta^3 + 3\delta^2\sigma - 3\delta\sigma^2) - d^+(v_1)\sigma^3]/6
\end{aligned}
\tag{4.2}
$$

It follows from the expressions for τ_v in (4.2) that $\tau_v \to \infty$ as $|a| \to 1$. Since $\upsilon(a)$ is a monotone function, if υ_1 is fixed and independent of a, we obtain a minimum $\tau_v = 2|\upsilon_1|^{1/2}$ corresponding to $a = 0$, $\upsilon(0) = 1$, that is $\delta = \sigma = |\upsilon_1|^{1/2}$ for $a = 0$.

The quantities τ_w, υ_1 and x_1 in (4.2) are defined in terms of w^0, u^0 and x^0 by (4.1). The function $u_v(t)$ changes sign in the second interval when $t = t_1 + \delta$, $\delta/\sigma = \upsilon^2$, which guarantees that the acceleration w will automatically vanish when $t = t_2$. A suitable choice of the sign of the control ($-$sign υ_1) and the length of the interval τ_v will reduce the velocity υ to zero at $t = t_2$; the value of x_2 (4.2) can be arbitrary (see Fig. 4).

We now consider the third, final stage of the control, completing the process whereby the phase point Q is brought from the state $Q_2(0, 0, x_2)$ to the origin $Q(t_3) = Q_f = (0, 0, 0)$. If $x_2 \neq 0$, the bang-bang control and the other parameters of the motion are defined by analogy with (4.1) and (4.2); the control has two switching points. We have the following governing relations

$$
\begin{aligned}
& t_2 < t \leqslant t_3 = \tau_w + \tau_v + \tau_x, \quad \tau_w = 2(\gamma + \varkappa) \\
& \varkappa = \gamma d^-(x_2)/d^+(x_2), \quad \gamma = (|x_2|/2)^{1/3}(d^+(x_2))^{2/3}\mu^{-1/3}(x_2) \\
& \mu = (d^-(x_2)(d^+(x_2))^2/6 + 2(d^-(x_2))^3 + 3(d^-(x_2))^2/d^+(x_2)) \\
& u = u_x(t) = -\operatorname{sign} x_2(h(t - t_2) - 2h(t - t_2 - \gamma) + 2h(t - t_2 - \tau_x/2 - \varkappa) - h(t - t_2 - \tau_x)) \\
& Q(t_3) = Q_f = (0, 0, 0)
\end{aligned}
\tag{4.3}
$$

Note that the control $u_x(t)$ is symmetrical about the midpoint of the interval $\tau_x/2 = \gamma + \kappa$. In the case of symmetrical constraints on the control (see Section 1, $a = 0$), we have $\gamma = \varkappa = \tau_x/4 = (|x_2|/2)^{1/3}$; by (4.1) and (4.2), the value of x_2 is $x_2 = x_1 + \upsilon_1 \tau_v$, where the length of the second step $\tau_v = 2|\upsilon_1|^{1/2}$ is determined by the value of the velocity υ_1 at its first point $\upsilon_1 = \upsilon^0 + w^0|w^0|/2$, and the displacement is $x_1 = x^0 + \upsilon^0|w^0| + w^{03}/3$. The resulting expression for $\tau_x(a)$ (4.3) indicates that $\tau_x \to \infty$ as $|a| \to 1$ (see above, the second stage) and for fixed x_2 independent of a, but the minimum is reached at a value $a^* = \operatorname{sign} x_2 (1 - \sqrt{(4/3)})$, $|a^*| \approx 1/6$. In addition, relations (4.1)–(4.3) imply the following estimate

$$t_3 = O(|w^0| + |\upsilon^0|^{2/3} + |x^0|^{1/3})$$

for asymptotically large values of $\|Q^0\|$. Note that the quantity t_3 of (4.3) defines an upper bound t_f^+ for the unknown t_f; see (3.2)–(3.4).

Figure 4 presents time histories of the phase variables w, υ, x and a "near-optimal" control u for the same initial data as before (see Sections 2 and 3). The desired time t_3 turned out to be t_3 = 7,06 for $a = 0$ and $t_3 = 5.07$ for $a = 1/2$. Figures 5 and 6 show projections of the phase trajectories onto the *WV* and *WX* planes analogous to those presented in Figs 2 and 3; an analysis and comparison of these curves would definitely be interesting.

4.2. Combined method of control. Compared with the coordinatewise mode of control considered above, according to relations (4.1)–(4.3), which give a very rough lower bound t_f^- and upper bound $t_f^+ = t_3$ for the minimum time t_f, it is fairly simple to implement a two-stage process which yields a sharper estimate of t_f: (1) time-optimal steering of the acceleration and velocity of the system to zero, (2) optimal steering to the terminal point Q_f. Thus, the first stage combines two stages of the "coordinatewise descent" control mode considered above and is reduced to the known exact solution [8] for the classical problem [1]: $\upsilon = u$, $u \leqslant u \leqslant u^+$.

The control at the first stage contains one switching point, determined by the time $t = \alpha$ at which the switching curve intersects the *WV* plane. There are explicit expressions for the optimal control and time characteristics of the control process at the first stage

$$\begin{aligned}
&0 < t \leqslant t_{1,2} = \tau_{w\upsilon} = \alpha + \beta, \quad 0 < \tau_{w\upsilon}(w^0, \upsilon^0) = t_f^- \leqslant t_f \\
&u_{w\upsilon}(t) = \{-\operatorname{sign}\Delta,\ 0 < t \leqslant \alpha;\ \operatorname{sign}\Delta,\ \alpha < t \leqslant \tau_{w\upsilon}\} \\
&\Delta = \upsilon^0 - V(w^0) \neq 0, \quad V(w^0) = -|w^0|w^0/d^-(w^0)/2 \\
&\alpha = w^0/\Lambda^- + \beta d^+(\Delta)/d^-(\Delta) \\
&\beta = [\upsilon^0 d^-(\Delta)/\Lambda^+ + w^{02}(1-a^2)^{-1}/2]^{1/2} \\
&\Lambda^\pm = d^\pm(\Delta) \operatorname{sign}\Delta
\end{aligned} \tag{4.4}$$

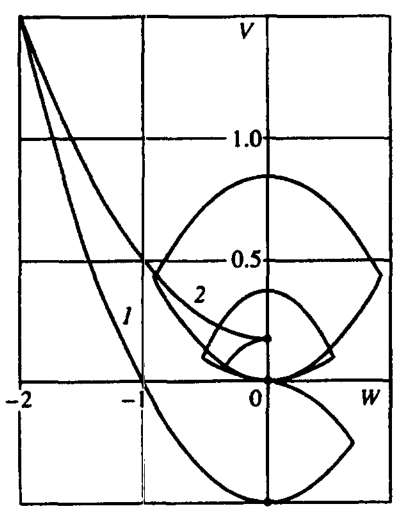

Fig. 5

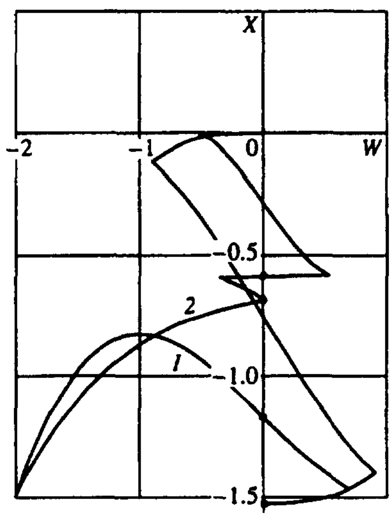

Fig. 6

It follows from (4.4) that for $a = 0$, we have $d^{\pm} = 1$ and the expressions for u_{wv}, τ_{wv}, α, β become the well-known expressions [1] for the case of symmetrical constraints. At the end of the first stage of the control, $w(\tau_{wv}) = \upsilon(\tau_{wv}) = 0$, and the variable $x(t)$ takes the value

$$\begin{gathered} x(\tau_{wv}) = x_{wv} = x_\alpha + v_\alpha\beta + w_\alpha\beta^2/2 + \Lambda^+\beta^3/6 \\ w_\alpha = w^0 - \Lambda^-\alpha, \quad v_\alpha = v^0 + w^0\alpha - \Lambda^-\alpha^2/2 \\ x_\alpha = x^0 + v^0\alpha + w^0\alpha^2/2 - \Lambda^-\alpha^3/6 \end{gathered} \tag{4.5}$$

where υ_α and w_α are the values of the variables υ and w at $t = \alpha$, that is, at the time the switching curve is reached according to (4.4), and $x_\alpha = x(\alpha)$ is the corresponding value of the variable x.

At the second stage, we have a time-optimal motion from the phase point $Q_{wv} = (0, 0, x_{wv})$ to the terminal point Q_f as in the third stage considered above in (4.3), where $x_2 = x_{wv}$. The desired total time is $t_3 = \tau_{wv} + \tau_x$, where $\tau_x = 4(|x_{wv}|/2)^{1/3}$. There are estimates for the time $t_3(t_3 \geqslant t_f^*)$ in terms of w^0, υ^0 and x^0, analogous to those presented for the simple three-stage mode of control; this value may be used as an upper limit for t_f^* in (3.2)–(3.4). As indicated in Section 4.1, one can construct graphs like those shown in Fig. 4 for the trajectories and control and like those shown in Figs 5 and 6 for the projections of the phase trajectories.

5. CONCLUSIONS AND POSSIBLE GENERALIZATIONS

Thus, Pontryagin's maximum principle and Fel'dbaum's theorem yield a highly efficient construction of a time-optimal feedback control as a synthesis for a third-order system. The feedback control problem has been successfully solved in an analytical form: the switching surface and curve are defined parametrically or explicitly. The choice of the sign of the bang-bang control reduces to verification of inequalities for relations defined by algebraic (power) functions.

The construction of an open-loop control requires a computation of the time intervals during which the control has a fixed sign and the optimal time, given initial values of the acceleration, velocity and coordinate. The optimal time is determined by solving a fourth-order algebraic equation whose coefficients depend on the initial data. It may be found numerically or by using Cardano's formula (see Section 3).

Along with the optimal mode, one can use extremely simple "near-optimal" control methods which are nearly globally optimal if the phase point is comparatively near the switching curve or surface (see Section 3).

It is possible to extend the above approach to the construction of controls in the open-loop or feedback form in the more general case of final conditions and asymmetrical constraints: $u^- \leqslant u \leqslant u^+$, where $u^- < 0, u^+ > 0$. The problem is described by an equation of the form

$$\dddot{x} = a + u', \quad a = (u^- + u^+)(u^+ - u^-)^{-1}, \quad |a| < 1$$

where a is a constant action, the control u' satisfies the inequality $|u'| \leqslant 1$ and the final conditions are

$$x(t_f) = y(t_f), \quad \dot{x}(t_f) = \dot{y}(t_f), \quad \ddot{x}(t_f) = \ddot{y}(t_f)$$

where $y(t)$ is a known function (for example, a second-order polynomial in t) describing uniformly accelerated motion of the terminal point. Time-optimal and near-time-optimal solutions are constructed by analogy with the procedure described above, on the basis of the maximum principle, and they are qualitatively of the same form. Investigation of the behaviour of the switching curve and switching surface, as well as the other characteristics of the control process, as functions of the parameter a and the phase vector Q, would need separate consideration.

For applications and the theory of optimal control, it is of great importance to take into account constantly acting perturbations, since the models considered above are highly idealized. For example, the perturbed system might have the form

$$\dddot{x} = \varepsilon f(x, \dot{x}, \ddot{x}) + u, \quad |\varepsilon| \ll 1, \quad (x, \dot{x}, \ddot{x}) \in D \subseteq R \tag{5.1}$$

where f is a fairly smooth function in D. In order to determine the switching curve and the switching surface of a bang-bang control u, approximate by the powers of the small parameter ε, one would need to develop a suitable procedure of the perturbation method, analogous to that described in [8]. A further natural generalization of system (5.1) would be to allow the function f to depend on a quasi-constant vector $z : \dot{z} = \varepsilon Z$, where z is the vector of the system parameters. The functions f and Z may depend on $x, \dot{x}, \ddot{x}$, and also on u and t.

It is of some interest to extend the technique to systems described by an nth-order equation of the form

$$\left[\prod_{j=1}^{n}\left(\frac{d}{dt}+\lambda_j\right)\right]x = u, \quad \operatorname{Im}\lambda_j = 0, \quad j = 1,\dots,n \geqslant 4 \tag{5.2}$$

which may also include constantly acting perturbations as in (5.1) and below. In particular, if system (5.2) corresponds to the situation $\lambda_j = 0$, that is, $x^{(n)} = u$, then the singular manifolds (curves, surfaces or hypersurfaces up to dimension $n - 1$) may be constructed successively as in Sections 1 and 2. Determination of the time characteristics for an open-loop control requires much larger computational resources. The construction of near-optimal modes ("coordinatewise descent") may be implemented by analogy with the various methods described in Section 3. The simplest mode would be successive reduction of the derivatives to zero, beginning with the $(n - 1)$th: $x^{(n-1)}, \dots, \dot{x}, x$. In particular, if $u^- = u^+$, the controls corresponding to these steps are constructed using Walsh functions [7]. One may also have combined modes, with time-optimal reduction to zero of the quantities $x^{(n-1)}, x^{(n-2)}$ at the first stage, as described previously [1], or of $x^{(n-1)}, x^{(n-2)}, x^{(n-3)}$ (see above), and so on.

We wish to thank F. L. Chernous'ko for his interest this and V. S. Patsko for constructive remarks. The research was supported financially by the Russian Foundation for Basic Research (99-01-00222).

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Translated by D.L.